

ON THE DIAGONAL HOOKS OF A SYMMETRIC PARTITION

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ABSTRACT. Using only a symmetric p -core and p -quotient, we give an explicit formula for the set of diagonal hook lengths of the associated symmetric partition.

1. INTRODUCTION

Suppose $\mathbb{N} = \{0, 1, \dots\}$ let $n \in \mathbb{N}$ and p be a prime. For standard definitions of a *partition* λ of n , its *dual* λ^* and *Young diagram* $[\lambda]$, a *hook* h_{ij} of $[\lambda]$ with *corner* (i, j) , the *hook length* $|h_{ij}|$, the *arm length* and *leg length* of h_{ij} , and a β -set X corresponding to λ , we refer readers to [1], [2], [3], [7].

A β -set X associated to a partition λ can be seen as a finite set of non-negative integers, represented by beads at integral points of the x -axis, i.e. a bead at position x for each x in X . Then X is a β -set to λ in the *extended sense* if we extend X infinitely in both directions with beads at all negative positions and spaces at all positions to the right of the position of the largest integer $x_k \in X$. In this interpretation, β -sets equivalent to X are the same infinite string of beads and spaces with the origin shifted a finite number of positions to the left. A *minimal* β -set X is an extended set where the first space is counted as 0. If X is a minimal β -set of λ , we define $|X|$ as the number of beads occurring to the right of the leftmost space.

Given a fixed integer p , we can arrange the nonnegative integers in an array of columns and consider the columns as runners of an *abacus* in order to represent X .

$$\begin{array}{ccccccc} 0 & 1 & \cdots & p-1 & & & \\ p & p+1 & & 2p-1 & & & \\ \vdots & & \ddots & & & & \\ mp & & & mp+p-1 & & & \end{array}$$

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The column containing γ for $0 \leq \gamma \leq p-1$ will be called the γ th runner of the abacus. The integers $\gamma, \gamma+p, \gamma+2p, \dots$ label corresponding positions $0, 1, 2, \dots$ on the γ th runner. Placing a bead at position x_j for each $x_j \in X$ gives the *abacus diagram* of X .

We deviate from standard notation by denoting by $h = (y, x]$ a hook arising from the β -set X of λ where $y \notin X$ and $x \in X$. We define the *hook length* of $(y, x]$ as $x - y$. Lemma describes a bijection between the set of hooks h_{ij} of the Young diagram $[\lambda]$ and the set of hooks $(y, x]$ of a β -set X of λ .

Lemma 1.1. Let λ be a partition of n and X a β -set of λ . A hook $h = (y, x]$ of X corresponds to the hook h_{ij} with corner node (i, j) in the Young diagram $[\lambda]$ where

$$i = |z \in \mathbb{N} : z \in X, z \geq x|$$

and

$$j = |z \in \mathbb{N} : z \notin X, z \leq y|.$$

Additionally, the leg length and arm length of h are $|z \in \mathbb{N} : z \in X, y < z < x|$ and $|z \in \mathbb{N} : z \notin X : y < z < x|$ respectively.

Proof. See pg. 180 in [1]. □

If $h = (y, x]$ is a hook of length p (henceforth a *p-hook*) of X then $\{y\} \cup X - \{x\}$ is a β -set for a partition λ_1 of $n - p$. We say that λ_1 and X_{λ_1} are achieved from λ and X respectively by removing a p -hook. In the opposite manner, we see that λ and X are gotten from λ_1 and X_{λ_1} respectively by adding a p -hook. Subsequently, the abacus diagram of X_{λ_1} is related to that of X by moving the bead at $x \in X$ up one position on the runner. Let X^0 be the unique β -set obtained from X by successively removing p -hooks until none are left. Thus X^0 will have no p -hooks. The partition λ^0 represented by X^0 is called the *p-core* of λ and is uniquely determined by λ . The abacus of the p -core λ^0 is obtained from the abacus of λ by pushing up the beads in each runner as high up as they can go (Theorem 2.7.16, [2]).

A hook $h = (y, x]$ of length divisible by p is said to be on the γ th runner if x is on the γ th runner. Then y is also on the γ th runner. In particular, hooks of length divisible by p are on the same runner if and only if they have the same residue modulo p . For $0 \leq \gamma \leq p-1$, let $X_\gamma = \{j : \gamma + jp \in X\}$ and let λ_γ be the partition represented by the β -set X_γ . Notice that this is the partition whose beads appear on the γ th runner of the abacus diagram of λ . Our convention will be that the *p-quotient* of λ is the sequence $(\lambda_0, \dots, \lambda_{p-1})$ obtained from X where $|X| \equiv 0 \pmod{p}$. We call X_γ the β -set of λ_γ *induced* by X .

A partition is *symmetric* if $\lambda = \lambda^*$. A p -quotient $(\lambda_0, \dots, \lambda_{p-1})$ is *symmetric* if $\lambda_i = \lambda_{p-i-1}^*$ where $0 \leq i \leq p-1$. The p -quotient and p -core of a partition λ and its dual λ^* are related in the following manner.

Lemma 1.2. Let X be a β -set for λ such that $|X| \equiv 0 \pmod{p}$. Let λ^* be the dual of λ , let $(\lambda^*)^0$ be the p -core of λ^* and let $\{\lambda^*\} = \{\lambda_0^*, \dots, \lambda_{p-1}^*\}$ be the p -quotient of λ^* . Then $(\lambda^*)^0 = (\lambda^0)^*$ and $(\lambda_\gamma)^* = \lambda_{p-1-\gamma}$ for $0 \leq \gamma \leq p-1$. Hence $\lambda = \lambda^*$ if and only if $\lambda^0 = (\lambda^0)^*$ and $(\lambda_\gamma)^* = \lambda_{p-1-\gamma}$.

Proof. See Proposition 3.5 in [7]. \square

Given a symmetric partition λ , we let $\delta(\lambda) = \{\delta_{ii}(\lambda)\}$ be the set of diagonal hooks where $h_{ii} = \delta_{ii}(\lambda)$. When there is no ambiguity we will set $\delta_{ii}(\lambda) = \delta_{ii}$. By abuse of notation δ_{ii} will also stand for the size of $\delta_{ii}(\lambda)$, and $\delta(\lambda)$ for the set of diagonal hook lengths of λ .

In this paper we give an explicit formula for $\delta(\lambda) = \{\delta_{ii}\}$ in terms of only the p -quotient and the p -core. These results are motivated by an ongoing study of the irrationalities of the character table of the alternating groups $A(n)$, which, by a classical result of Frobenius, arise from the diagonal hook lengths of symmetric partitions of n . In particular they assist in verifying that a recent refinement of McKay's conjecture by Navarro [6] involving Galois automorphisms holds for $A(n)$ in special cases [5].

2. BISEQUENCES AND DIAGONAL HOOKS

A *bisequence*

$$(\alpha_1, \alpha_2, \dots, \alpha_t | \beta_1, \beta_2, \dots, \beta_t)$$

will be an ordered pair of strictly decreasing sequences of non-negative integers

$$\begin{array}{c} (\alpha_1, \alpha_2, \dots, \alpha_t) \\ (\beta_1, \beta_2, \dots, \beta_t) \end{array}$$

of the same length t . For example let α_i and β_i be the leg and arm lengths of $\delta_{ii} \in \delta(\lambda)$. Then the sequences $(\alpha_1, \alpha_2, \dots, \alpha_t)$ and $(\beta_1, \beta_2, \dots, \beta_t)$, are strictly decreasing. Hence we may define the bisequence

$$D(\lambda) = (\alpha_1, \alpha_2, \dots, \alpha_t | \beta_1, \beta_2, \dots, \beta_t).$$

We define the *components* of $D(\lambda)$ to be $D(\lambda)_L = (\alpha_1, \alpha_2, \dots, \alpha_t)$ and $D(\lambda)_R = (\beta_1, \beta_2, \dots, \beta_t)$. An element of $D(\lambda)$ is an ordered pair $(\alpha_i | \beta_i)$ for some i and corresponds to a diagonal hook δ_{ii} . Then $|D(\lambda)|$ is the number of such pairs, and equals t . Note that $|D(\lambda)_R|$ and

$|D(\lambda)_L|$ (the number of arm lengths and leg lengths of the diagonal hooks respectively) both equal t as well. The *dual* of $D(\lambda)$ is

$$D(\lambda)^* = (\beta_1, \beta_2, \dots, \beta_t | \alpha_1, \alpha_2, \dots, \alpha_t).$$

Clearly $D(\lambda)^* = D(\lambda^*)$. If λ is symmetric then $D(\lambda)_L = D(\lambda)_R$. We attach to $D(\lambda)$ a p -tuple $D'(\lambda)$ of bisequences.

Definition 2.1. Let $D'(\lambda) = (D_0(\lambda), \dots, D_{p-1}(\lambda))$ where $D_\gamma(\lambda)$ is defined as follows.

- (1) If $\alpha = \gamma + mp \in D(\lambda)_L$, $0 \leq \gamma \leq p-1$ and $m \geq 0$, we put m in $D_{p-1-\gamma}(\lambda)_L$.
- (2) If $\beta = \gamma + mp \in D(\lambda)_R$, $0 \leq \gamma \leq p-1$ and $m \geq 0$, we put m in $D_\gamma(\lambda)_R$.

$D'(\lambda)$ is called the *p-quotient* of $D(\lambda)$.

From this definition, given $D'(\lambda)$, we can obtain $D(\lambda)$. At the moment, it is not clear that for each γ the sequences in $D_\gamma(\lambda)_L$ and $D_\gamma(\lambda)_R$ have the same length. This will be shown to be true when $\lambda^0 = \emptyset$ in Theorem 5.2.

3. $\theta(\lambda)$ AND DIAGONAL HOOKS

The diagonal hooks δ_{ii} of λ correspond to the following hooks of β -set $X = \{x_1, \dots, x_k\}$. The largest hook δ_{11} corresponds to $(y_1, x_k]$, where y_1 is the position of the smallest space i.e. the minimal positive integer not included in X . By removing δ_{11} (that is, by moving the bead at position x_k to the space y_1) then δ_{22} corresponds to the largest hook of λ^\vee , and so on. Thus the diagonal hooks correspond to the nested hooks starting with the longest hook in X , then the longest hook contained strictly within that longest hook, and so on.

Let λ^\vee be the partition obtained from λ by removing δ_{11} . Then X^\vee is the induced β -set of λ^\vee .

Proposition 3.1. Suppose λ is a partition of n and let X be a β -set for λ . Then there exists a half-integer $\theta(\lambda)$ such that the number of beads to the right of $\theta(\lambda)$ equals the number of spaces to the left of $\theta(\lambda)$.

Proof. Let θ be the point at the half-integer just to the left of the smallest space of X . So there are 0 spaces to the left of θ and a finite number of beads to the right. Next move θ a unit distance to the right, that is, to the next half-integer on the right. One and only one of the following happens: Either the number of spaces to the left of θ increases by one or the number of beads to the right decreases by

one. So by iterating this process we reach a point where the number of spaces to the left of θ is equal to the number of beads to the right of θ . That point is then $\theta(\lambda)$. \square

Let X be a β -set for a partition λ (not necessarily symmetric), with maximal element $x_k \in X$. Let X_+ be the subset of beads to the right of $\theta(\lambda)$ and let X_- be the subset of spaces to the left of $\theta(\lambda)$. We index the elements of $X_+ = \{y'_i : i \leq r\}$ so that $y'_r < \cdots < y'_1$ with y'_1 as the largest bead. Correspondingly, we index the elements of $X_- = \{y_i : i \leq r\}$ so that $y_1 < \cdots < y_r$ with y_1 as the smallest space. In particular, $y'_i - y_i = \alpha_i + \beta_i + 1$ for all $1 \leq i \leq r$ since the length of the hook is one plus the sum of its leg and arm lengths. This relation holds whether or not X is a minimal β -set since y'_i, y_i and $\theta(\lambda)$ shift by the same amount when X is shifted, whereas α_i and β_i do not change. In particular the β -set X gotten from removing the largest diagonal hook of X can be used so that $\theta(\lambda^\vee) = \theta(\lambda)$ holds. Hence we have the following.

Lemma 3.2. Suppose λ' and λ are partitions such that $|\lambda'| < |\lambda|$ and λ' can be obtained from λ by removing a sequence of hooks from λ . Then $\theta(\lambda) = \theta(\lambda')$.

Proposition 3.3. X_- and X_+ correspond to $D(\lambda)_L$ and $D(\lambda)_R$ in the following manner. Let $\alpha_i \in D(\lambda)_L$ and $\beta_i \in D(\lambda)_R$. Then for each $y'_i \in X_+$ and $y_i \in X_-$ we have

- (1) $y_i = \theta(\lambda) - \frac{1}{2} - \alpha_i$.
- (2) $y'_i = \theta(\lambda) + \frac{1}{2} + \beta_i$

Proof. We proceed by induction on $s = |D(\lambda)|$. Suppose $s = 1$. Then $D(\lambda) = \{\alpha|\beta\}$ and $X = \{1, 2, \dots, m-1, m, t_1\}$ so $\theta(\lambda) = m + \frac{1}{2}$. Recall α and β are the number of beads and spaces respectively in the interval $(0, t_1)$, and hence $\alpha = \theta(\lambda) - \frac{1}{2}$ and $\beta = t - \theta(\lambda) - \frac{1}{2}$.

Consider λ where $|D(\lambda)| = s$ and let λ^\vee be the partition obtained by removing h_{11} from λ . Then, by induction and Lemma 3.2, when $|D(\lambda^\vee)| = s - 1$ we have that

$$y_i = \theta(\lambda) - \frac{1}{2} - \alpha_i, \quad y'_i = \theta(\lambda) + \frac{1}{2} + \beta_i$$

for $2 \leq i \leq r$. In particular, $y_2 = \theta(\lambda) - \frac{1}{2} - \alpha_2$ and $y'_2 = \theta(\lambda) + \frac{1}{2} + \beta_2$. But $y_2 - y_1$ is one plus the number of beads between y_1 and y_2 , which is precisely the difference $\alpha_1 - \alpha_2$ by Lemma 1.1. These formulas imply $y_1 = \theta(\lambda) - \frac{1}{2} - \alpha_1$ and $y'_1 = \theta(\lambda) + \frac{1}{2} + \beta_1$. \square

Suppose λ is symmetric. Then the diagonal hook lengths δ_{ii} are necessarily odd. Then X has a *axis of symmetry* $\theta(\lambda)$ where beads and

spaces on one side are reflected respectively into spaces and beads on the other side.

Corollary 3.4. Suppose λ is a symmetric partition and X is a β -set in the extended sense for λ . Then there exists an axis of symmetry $\theta(\lambda)$ at a half-integer such that beads and spaces in X to the right of $\theta(\lambda)$ are reflected respectively to spaces and beads in X to the left of $\theta(\lambda)$.

Proof. Follows from Proposition 3.1 and Proposition 3.3. \square

Lemma 3.5. Suppose λ is symmetric with empty p -core. Then the number of beads to the right of $\theta(\lambda)$ on the γ th runner is the same as the number of empty positions to the left of $\theta(\lambda)$ on the γ th runner.

Proof. Let X be a β -set for λ and let X^0 be the β -set obtained from X by sliding all beads to the top of the p -abacus of X . Then X^0 is a β -set for λ^0 . Since X is symmetric about $\theta(\lambda)$, and X^0 is obtained from X by removing successive diagonal hooks, so X^0 is also symmetric about $\theta(\lambda)$. But X^0 represents $\lambda^0 = \emptyset$, so it consists of $\{0, 1, 2, \dots, t\}$. Then it is clear that symmetry about $\theta(\lambda)$ is possible only if $\theta(\lambda) = t + 1/2$. Thus all beads of X^0 are left of the axis and all spaces of X^0 are right of the axis. Hence, all beads on the γ -runner right of the axis must be accommodated by spaces on the γ -runner left of the axis. \square

We recall that the labeling of the beads on the γ th runner of the abacus diagram of λ gives a β -set of the partition λ_γ . Note that each λ_γ has an axis of symmetry $\theta(\lambda_\gamma)$. Let X^0 be a β -set of λ^0 where $|X| \equiv 0 \pmod{p}$. In particular, we suppose $|X| = mp$ for some nonnegative integer m .

Lemma 3.6. If λ has empty p -core, then $\theta(\lambda_\gamma) = \theta(\lambda_{\gamma'})$ for all $0 \leq \gamma, \gamma' \leq p-1$. In particular, if $|X| = mp$, then $\theta(\lambda_\gamma) = \theta(\lambda_{\gamma''}) = m - \frac{1}{2}$.

Proof. Since λ has empty p -core the β -set for λ^0 is $X = \{0, 1, \dots, mp-1\}$ for some m . Then the abacus diagram will consist of (from north-to-south) m rows of beads followed by rows of empty spaces. On each runner one begins counting at 0, hence $\theta(\lambda_\gamma^0) = \theta(\lambda_{\gamma'}^0) = m - \frac{1}{2}$. By Lemma 3.2, $\theta(\lambda_\gamma^0) = \theta(\lambda_\gamma)$ and $\theta(\lambda_{\gamma'}^0) = \theta(\lambda_{\gamma'})$. The result follows. \square

We will need the following relation in Section 8.

Lemma 3.7. Suppose λ has empty p -core. Then

$$p(\theta(\lambda_\gamma) + \frac{1}{2}) = \theta(\lambda) + \frac{1}{2}$$

for all $0 \leq \gamma \leq p-1$.

Proof. Since λ has an empty p -core we have $X^0 = \{0, 1, 2, \dots, mp-1\}$ for some m . Hence $\theta(\lambda) = mp - \frac{1}{2}$. Since mp is the total number of beads in X , we have

$$p(m-1+1) = mp - \frac{1}{2} + \frac{1}{2}$$

which implies $p(\theta(\lambda_\gamma) + \frac{1}{2}) = \theta(\lambda) + \frac{1}{2}$ by Lemma 3.6. \square

Let λ be such that $\lambda^0 \neq \emptyset$ and let $\bar{\lambda}$ be such that $\bar{\lambda} = \emptyset$ but $\bar{\lambda}_i = \lambda_i$ for $0 \leq i \leq p-1$.

Corollary 3.8. For all $0 \leq \gamma \leq p-1$ we have

$$p(\theta(\bar{\lambda}_\gamma) + \frac{1}{2}) = \theta(\lambda) + \frac{1}{2}.$$

Proof. By Lemma 3.2, $\theta(\bar{\lambda}) = \theta(\lambda)$. The result then follows from Lemma 3.7. \square

We use Corollary 3.8 to offer an interpretation of $D'(\lambda)$. Suppose $\alpha_i \in D(\lambda)_{L,\gamma}$ so $\alpha_i = \gamma + \eta_i p$. Then $\gamma + \eta_i p = \theta(\lambda) - \frac{1}{2} - y_i$ by Proposition 3.3. By Corollary 3.8

$$\gamma + \eta_i p = p(\theta(\bar{\lambda}_\gamma) - \frac{1}{2}) + p - 1 - y_i.$$

Suppose $y_i = \gamma^* + \bar{y}_i p$. Then $\bar{y}_i = \theta(\bar{\lambda}_\gamma) - \eta_i - \frac{1}{2}$.

Now suppose $\beta_i \in D(\lambda)_{R,\gamma}$ so $\beta_i = \gamma + \eta_i p$. Then $\gamma + \eta_i p = y'_i - \theta(\lambda) - \frac{1}{2}$. Then

$$\gamma + \eta_i p = y'_i - p(\theta(\lambda_\gamma) + \frac{1}{2}).$$

by Proposition 3.3. Hence $y'_i - \gamma = p(\theta(\lambda_\gamma) + \frac{1}{2} + \eta_i)$. Suppose $y'_i = \gamma + \bar{y}'_i p$. Then $\bar{y}'_i = \theta(\lambda_\gamma) + \frac{1}{2} + \eta_i$. Hence $D'(\lambda) = \{D_\gamma(\lambda)\}_{0 \leq \gamma \leq p-1}$ can be expressed as distances of the beads and spaces of the corresponding X_γ from each $\theta(\bar{\lambda}_\gamma)$. We will use this observation in Section 8.

4. PAIRS OF STRADDLING OR NON-STRADDLING p -HOOKS

If $(x', x]$ is a diagonal hook of X corresponding to a symmetric λ , we call x' the *opposite position* of x . Given two diagonal hooks $(x', x]$ and $(y', y]$ where $x < y$, we call the (non-diagonal) hooks $(y', x]$ and $(x', y]$ *opposite hooks*. Conversely, given opposite non-diagonal hooks $(y', x]$ and $(x', y]$ with $x < y$ we get diagonal hooks $(y', y]$ and $(x', x]$.

Lemma 4.1. Suppose λ is symmetric and let X be a β -set for λ with $|X| \equiv 0 \pmod{p}$. Let $(x', x]$ be a diagonal hook of X . Then $x' \equiv p-1-\gamma \pmod{p}$ if and only if $x \equiv \gamma \pmod{p}$.

Proof. By symmetry around $\theta(\lambda)$, the number of beads and empty positions below the axis is $|X|$. Hence $\theta(\lambda) - \frac{1}{2} \equiv p - 1 \pmod{p}$. Since x' and x are equidistant from $\theta(\lambda)$, then $x' \equiv p - 1 - x \pmod{p}$. \square

Suppose we want to reduce a symmetric partition λ of n to a symmetric partition λ of $n - p$ by removing one p -hook. There is one way of doing so.

- (1) (*The single hook case*) Then p -hook $h = (y, x]$ is a diagonal hook $(x', x]$ where $x - x' = p$.

Suppose we want to reduce a symmetric partition λ of n to a symmetric partition λ' of $n - 2p$ by removing two p -hooks. By removing two opposite p -hooks $h = (y, x]$ and $h' = (x', y']$ where $h \neq h'$. There are two cases, the *non-straddling case*, in which $x' < y' < \theta(\lambda) < y < x$, and the *straddling case*, in which $x' < y < \theta(\lambda) < y' < x$.

- (1) (*The non-straddling case*). Suppose h is completely to the right of $\theta(\lambda)$. Then removing h and h' is equivalent to replacing a diagonal hook $(x', x]$ with $(x' + p, x - p]$.
- (2) (*The straddling case*). Suppose that h and h' straddle $\theta(\lambda)$. Then removing h and h' is equivalent to removing two diagonal hooks $(x', x]$ and $(y, y']$ where $x - x' + y' - y = 2p$.

Suppose $h = (y, x]$ and $h' = (x', y']$ are non-straddling opposite p -hooks of λ (as in Figure 2). Without loss of generality, $x > y'$. Let $h = h_{ij}$, i.e. have corner (i, j) in $[\lambda]$. Then the corner of h is on the arm of some diagonal hook. Since

$$|\{z \in \mathbb{N} : z \notin X, z \leq y\}| > |\{z \in \mathbb{N} : z \in X, z \geq x\}|$$

by Lemma 1.1, we have $j > i$. Thus, if $x = \gamma + kp$, where $0 \leq \gamma \leq p - 1$ and $k \geq 0$, then $y \notin X$ such that $y = \gamma + (k - 1)p$. Consequently, $h = (y, x]$ of λ corresponds to some hook $(k - 1, k]$ of λ_γ on the γ th runner of the p -abacus. We give the exact coordinates (i_h, j_h) on the Young diagram $[\lambda_\gamma]$ corresponding to $(k - 1, k]$ when λ has empty p -core. Define

$$\begin{aligned} A &= \{z \equiv \gamma \pmod{p}, z \in X : z \geq x\} \\ B &= \{z \equiv \gamma \pmod{p}, z \notin X : \theta(\lambda) < z \leq y\} \\ C &= \{z \equiv -1 - \gamma \pmod{p}, z \in X : z > \theta(\lambda)\}. \end{aligned}$$

Let $|A| = a$, $|B| = b$ and $|C| = c$. By construction, $i_h = a$.

By Proposition 3.1, λ_γ has an axis $\theta(\lambda_\gamma)$ that is a half-integer such that the number of beads above $\theta(\lambda_\gamma)$ is the same as the number of spaces below.

Lemma 4.2. Suppose λ has empty p -core and $Y = \{w_1, \dots, w_j\}$ is the induced β -set for λ_γ . Let k be an integer such that $0 \leq k \leq w_j$. Then we have the following.

- (1) If $k < \theta(\lambda_\gamma)$ then $(p - \gamma - 1) + kp < \theta(\lambda)$
- (2) If $k > \theta(\lambda_\gamma)$ then $\gamma + kp > \theta(\lambda)$.

Proof. Follows by Proposition 3.7 and the definition of the p -quotient. \square

Lemma 4.3. Suppose λ is a symmetric partition with empty p -core. Consider the p -hook $h = (y, x]$. Let (i_h, j_h) be the coordinates of the corresponding 1-hook of $[\lambda_\gamma]$ for some fixed γ . Then

$$j_h = b + c$$

if and only if h is completely to the right of $\theta(\lambda)$.

Proof. Suppose $j_h = b + c$. It is clear that h is completely to the right of $\theta(\lambda)$. Suppose h is completely to the right of $\theta(\lambda)$. Since λ is symmetric, we know by Lemma 4.1 that C corresponds bijectively to the set $\{y' \in \mathbb{N}, y' \notin X, y < \theta(\lambda) : y' \equiv \gamma \pmod{p}\}$. Hence c is also the number of empty positions less than $\theta(\lambda)$ of residue $\gamma \pmod{p}$. By Lemma 3.5, b is the number of empty positions between $\theta(\lambda)$ (and including) y with residue $\gamma \pmod{p}$. This follows since λ has empty p -core. Hence $b + c$ is the total number of empty positions below and including y with residue $\gamma \pmod{p}$. Then, by Lemma 1.1, we are done. \square

Lemma 4.4. Suppose λ is symmetric with empty p -core. Consider the p -hook $h = (y, x]$. If h is completely to the right of $\theta(\lambda)$ then

$$a \leq c$$

Proof. By Lemma 4.1, we have that c is the number of empty positions less than $\theta(\lambda)$ that have residue $\gamma \pmod{p}$. By Lemma 3.5, since λ has empty p -core, c must be equal to the number of $z \in X$ such that $z > \theta(\lambda)$ and $z \equiv \gamma \pmod{p}$. Since $x > \theta(\lambda)$ and $A = \{z = \gamma + jp, z \in X : z \geq x\}$, we have $a \leq c$. \square

Proposition 4.5. A hook $(k - 1, k]$ of size 1 on λ_γ corresponds to the p -hook $h = (y, x]$ on λ where y and x are completely to the right (resp. left) of $\theta(\lambda)$ if and only if $(k - 1, k]$ occurs on an arm (resp. leg) of $[\lambda_\gamma]$.

Proof. Suppose h is to the right of $\theta(\lambda)$. The coordinates of $(k - 1, k]$ on the Young diagram $[\lambda_\gamma]$ are (i_h, j_h) where $i_h = a$ (by definition) and $j_h = b + c$ by Lemma 4.3. Since $y \in B$, $|B| \neq 0$ and we have $a < b + c$,

since $a \leq c$ by Lemma 4.4. It follows that $i_h < j_h$. Hence $(k-1, k]$ is a 1-hook on the arm of $[\lambda_\gamma]$.

Suppose $(k-1, k]$ is a 1-hook on the arm of $[\lambda_\gamma]$. Clearly $\theta(\lambda_\gamma) < k-1$. Hence $\theta(\lambda) < y = \gamma + (k-1)p$ by Lemma 4.2. Since $y < x$, h is completely to the right of $\theta(\lambda)$.

Suppose $(k-1, k]$ is a 1-hook on λ_γ corresponding to a p -hook $h = (y, x]$ completely to the left of $\theta(\lambda)$. Then h can be viewed as a hook completely to the right of $\theta(\lambda^*)$. Hence, by the argument above, it corresponds to a 1-hook on the arm of λ_{γ^*} . Taking the dual again, h corresponds to a 1-hook on the leg of λ_γ . \square

Proposition 4.6. Let λ be a symmetric partition with empty p -core. Let $\gamma \in [0, p-1]$. Then a hook $h = (k-1, k]$ of size 1 on λ_γ corresponds to the p -hook $h = (y, x]$ of λ where y is to the left of $\theta(\lambda)$ and x is to the right of $\theta(\lambda)$ if and only if $(k-1, k]$ is a diagonal hook of $[\lambda_\gamma]$.

Proof. Consider a pair $h = (y, x]$ and $h' = (x', y']$ of straddling p -hooks. Without loss of generality, we may suppose $x' < y < y' < x$. Set $y' = \gamma + jp$ and $x = (p-1-\gamma) + kp$. So $x' = \gamma + (j-1)p$ and $y = (p-1-\gamma) + (k-1)p$. Suppose $E = \{z \in X, z = \gamma \pmod{p}, z \geq y'\}$ and $F = \{z' \notin X, z' = \gamma \pmod{p} : z' \leq x'\}$. Let $|E| = e$ and $|F| = f$. By Lemma 1.1 the coordinates of the hook $(k-1, k]$ on the Young diagram $[\lambda_\gamma]$ are $(i_h, j_h) = (e, f)$. The inequality $x' < y' < x$ is equivalent to $(k-1)p < 2\gamma - p + 1 + jp < kp$. If $\gamma = \frac{p-1}{2}$, then $h = h'$, which is impossible. Hence we only consider $\gamma \neq \frac{p-1}{2}$. If $0 \leq \gamma < (p-1)/2$, it follows that $j = k$ and $\theta(\lambda) = kp - 1/2$. If $(p-1)/2 < \gamma \leq p-1$, it follows $j = k-1$ and $\theta(\lambda) = jp + (p-1)/2$. Since $\theta(\lambda)$ is a half-integer, the second case is impossible and $0 \leq \gamma < (p-1)/2$. Now define $E' = \{z \in X : z = \gamma \pmod{p}, z > \theta(\lambda)\}$ and $F' = \{z' \in \mathbb{N} : z' \notin X, z' = \gamma \pmod{p}, z' < \theta(\lambda)\}$, so $|E'| = |F'|$ by Lemma 3.5. Since $y < \theta(\lambda) < x$ and $x - y = p$, we have $\{z \in X : z = \gamma \pmod{p}, \theta(\lambda) < z < x\} = \{z' \notin X : z' = \gamma \pmod{p}, y < z' < \theta(\lambda)\} = \emptyset$. Thus $E' = E$ and $F' = F$ and we are done.

Suppose $(k-1, k]$ is a hook of size 1 on the diagonal of $[\lambda_\gamma]$. Then $k-1 < \theta(\lambda_\gamma) < k$. Hence $(p-1-\gamma + (k-1)p, \gamma + kp] = (y, x]$ straddles $\theta(\lambda)$ by Lemma 4.2. \square

5. $D'(\lambda)$ CONCENTRATED AT ONE OR TWO PLACES

Suppose λ is a symmetric partition. We say $D(\lambda)$ is *concentrated* at $\{\gamma, \gamma^*\}$ if $D_i(\lambda) \neq \emptyset$ for $i \in \{\gamma, \gamma^*\}$ and $D_i(\lambda) = \emptyset$ otherwise.

Lemma 5.1. Suppose λ and λ' are distinct partitions such that $D(\lambda)$ is concentrated at $\{\gamma, \gamma^*\}$ and $D(\lambda')$ is concentrated at $\{\gamma', \gamma'^*\}$ where $\gamma \neq \gamma'$. If $\{\gamma, \gamma^*\} \neq \{\gamma', \gamma'^*\}$ then $D(\lambda) \cap D(\lambda') = \emptyset$, that is, no diagonal hook length of λ equals a diagonal hook length of λ' .

Proof. Suppose not. Then there exists $\alpha \in D(\lambda)_L$ and $\alpha' \in D(\lambda')_L$ such that $\alpha = \alpha'$. But $\alpha = \gamma + mp$ and $\alpha' = \gamma' + m'p$ so that $\gamma = \gamma'$. This is impossible. \square

Suppose λ and λ' are symmetric partitions such that $D(\lambda) \cap D(\lambda') = \emptyset$. Define $\lambda + \lambda'$ to be the symmetric partition such that $D(\lambda + \lambda') = D(\lambda) \cup D(\lambda')$. In particular, we can form $\lambda + \lambda'$ whenever λ and λ' are concentrated on disjoint sets.

Theorem 5.2. Let λ be symmetric with empty p -core such that $D'(\lambda)$ is concentrated at $\{\gamma, \gamma^*\}$ where $\gamma \neq \gamma^*$. Then

- (1) $D'(\lambda)$ is a p -tuple of bisequences, that is, for each γ , $D_\gamma(\lambda)_L$ and $D_\gamma(\lambda)_R$ are of equal lengths.
- (2) For each γ , $D_\gamma(\lambda) = D(\lambda_\gamma)$ and $D_{\gamma^*}(\lambda) = D(\lambda_{\gamma^*})$, where λ_γ and λ_{γ^*} are the γ th and γ^* th components of the p -quotient of λ .
- (3) Suppose $D(\lambda_\gamma) = (\sigma_1, \dots, \sigma_w | \tau_1, \dots, \tau_w)$. Then

$$D(\lambda) = (\alpha_1, \dots, \alpha_{2w} | \alpha_1, \dots, \alpha_{2w})$$

$$\text{where } \{\alpha_1, \dots, \alpha_{2w}\} =$$

$$\{\gamma^* + \sigma_i p, \gamma + \tau_i p : 1 \leq i \leq w\}.$$

Proof. By induction on $|\lambda|$. The minimal case is $|\lambda| = 2p$ where $D(\lambda) = (p-1-\gamma, \gamma | p-1-\gamma, \gamma)$. Then λ is comprised of just two opposite p -hooks. Hence $|D_\gamma(\lambda)_R| = |D_\gamma(\lambda)_L| = 1$ and part (1) follows. By definition, $D_\gamma(\lambda) = (0|0)$ and $D_{p-1-\gamma}(\lambda) = (0|0)$. Since $\lambda_\gamma = (1)$ and $\lambda_{\gamma^*} = (1)$, part (2) follows. Part (3) follows since $D(\lambda) = (p-1-\gamma, \gamma | p-1-\gamma, \gamma)$. Now suppose $|\lambda| = n > 2p$. By induction, we assume that the theorem holds for all partitions λ such that $|\lambda| < n$. Consider $|\lambda| = n$. Let h, h' be opposite p -hooks in λ and let λ^\vee be the symmetric partition gotten from removing h and h' . Following the discussion preceding Lemma 4.3, there are two cases.

Case 1: (The non-straddling case) Here one obtains $D(\lambda^\vee)$ from $D(\lambda)$ by replacing an element $(\alpha | \alpha)$ by $(\alpha - p | \alpha - p)$ where $\alpha - p \geq 0$. Then $D_\gamma(\lambda)_L$ and $D_\gamma(\lambda^\vee)_L$ are the same except for some $\sigma_\mu \in D_\gamma(\lambda)_L$ which is replaced by $\sigma_\mu - 1$. By symmetry, σ_μ is replaced by $\sigma_\mu - 1$ resulting in $D_{\gamma^*}(\lambda')_R$. We prove that parts (1), (2), and (3) hold.

- (1) By induction $D_\gamma(\lambda^\vee)$ and $D_{\gamma^*}(\lambda^\vee)$ are both bisequences with components of equal length. Hence the same is true for $D_\gamma(\lambda)$ and $D_{\gamma^*}(\lambda)$.

- (2) By induction $D_\gamma(\lambda^\vee) = D(\lambda_\gamma^\vee)$. By Proposition 4.5, λ_γ is obtained from λ_γ^\vee by adding a hook of size 1 to both the leg length of the diagonal hook corresponding to $(\sigma_\mu - 1 | \tau_\mu) \in D(\lambda_\gamma')$ and to the arm length of the diagonal hook $(\tau_\mu | \sigma_\mu - 1) \in D(\lambda_{\gamma^*}^\vee)$. Hence $D_\gamma(\lambda) = D(\lambda_\gamma)$ and $D_{\gamma^*}(\lambda) = D(\lambda_{\gamma^*})$.
- (3) Given

$$D(\lambda_\gamma^\vee) = (\sigma_1, \dots, \sigma_\mu - 1, \dots, \sigma_w | \tau_1, \dots, \tau_i, \dots, \tau_w)$$

$$D(\lambda_{p-1-\gamma}^\vee) = (\tau_1, \dots, \tau_i, \dots, \tau_w | \sigma_1, \dots, \sigma_\mu - 1, \dots, \sigma_w)$$

we have by induction that

$$D(\lambda^\vee) = (\dots \alpha'_i, \alpha''_i \dots | \dots \alpha'_i, \alpha''_i \dots)$$

where

$$\alpha'_i = (p - 1 - \gamma) + \sigma_i p \quad \alpha''_i = \gamma + \tau_i p$$

$1 \leq i \leq w$ and $i \neq \mu$. When $i = \mu$, then

$$\alpha'_\mu = (p - 1 - \gamma) + (\sigma_\mu - 1)p \quad \alpha''_\mu = \gamma + \tau_\mu p$$

It is clear by replacing $\sigma_\mu - 1$ by σ_μ , that the desired formula for $D(\lambda)$ is obtained.

Case 2:(The straddling case) Here $D(\lambda^\vee)$ one obtains $D(\lambda)$ from by removing $(\alpha|\alpha)$ and $(\beta|\beta)$ where $\alpha + \beta + 1 = p$ (assume without loss of generality that $\alpha > \beta$). Then $D(\lambda^\vee)$ is also concentrated at $\{\gamma, \gamma^*\}$. The relation $\alpha + \beta + 1 = p$ implies that if $\alpha = \gamma$, then $\beta = p - 1 - \gamma$. Thus $(\alpha|\alpha)$ contributes a term 0 to $D_\gamma(\lambda)_L$ and a 0 to $D_{\gamma^*}(\lambda)_R$. Likewise $(\beta|\beta)$ contributes a term 0 to $D_\gamma(\lambda)_R$ and a 0 to $D_{\gamma^*}(\lambda)_L$. We prove that parts (1), (2), and (3) hold.

- (1) By induction $D_\gamma(\lambda^\vee)$ is a bisequence with components of equal length. Hence $D_\gamma(\lambda)$.
- (2) By induction $D_\gamma(\lambda^\vee) = D(\lambda_\gamma^\vee)$ and $D_{p-1-\gamma}(\lambda^\vee) = D(\lambda_{\gamma^*}^\vee)$. Now we re-attach to λ^\vee the diagonal hooks corresponding to $(\alpha|\alpha)$ and $(\beta|\beta)$, where $\alpha + \beta + 1 = p$. This is equivalent to adjoining $(0|0)$ to both $D_\gamma(\lambda)$ and $D_{\gamma^*}(\lambda)$. The effect on the partitions λ_γ^\vee and $\lambda_{\gamma^*}^\vee$ will be adding a diagonal node of size 1 to each, by Proposition 4.6. Hence $D_\gamma(\lambda) = D(\lambda_\gamma)$ and $D_{\gamma^*}(\lambda) = D(\lambda_{\gamma^*})$.
- (3) Given

$$\begin{aligned} D(\lambda_\gamma^\vee) &= (\sigma_1, \dots, \sigma_{w-1} | \tau_1, \dots, \tau_{w-1}) \\ D(\lambda_{\gamma^*}^\vee) &= (\tau_1, \dots, \tau_{w-1} | \sigma_1, \dots, \sigma_{w-1}) \end{aligned}$$

then by induction

$$D(\lambda^\vee) = (\dots \alpha'_i, \alpha''_i \dots | \dots \alpha'_i, \alpha''_i \dots)$$

where

$$\alpha'_i = (p - 1 - \gamma) + \sigma_i p \quad \alpha''_i = \gamma + \tau_i p$$

and $1 \leq i \leq w - 1$. Now attaching $(0|0)$ to $D(\lambda_\gamma^\vee)$ and $(0|0)$ to $D(\lambda_{\gamma^*}^\vee)$, is equivalent to adjoining both $p - 1 - \gamma$ and γ to both $D(\lambda)_R$ and $D(\lambda)_L$. Hence the desired formula for $D(\lambda)$ is obtained. \square

Corollary 5.3. Suppose λ is symmetric with empty p -core and $D(\lambda)$ is concentrated at $\{\gamma, \gamma^* : \gamma \neq \gamma^*\}$ and $D_\gamma(\lambda) = (\sigma_1, \dots, \sigma_w | \tau_1, \dots, \tau_w)$. Then

$$\delta(\lambda) = \cup_i \{2(\sigma_i + 1)p - 2\gamma - 1, 2\tau_i p + 2\gamma + 1\}$$

Proof. Follows from part 3 of Theorem 5.2. \square

Example 5.4. Let $p = 5$. Suppose the 5-quotient is concentrated at $\{\gamma, \gamma^*\} = \{0, 4\}$, where $\lambda_0 = (6^2, 2)$ and $\lambda_4 = (3^2, 2^3)$. Then $D_0(\lambda) = (2, 1|5, 4)$ and $D_4(\lambda) = (5, 4|2, 1)$, $D(\lambda) = (25, 20, 14, 9|25, 20, 14, 9)$ and $\delta(\lambda) = (51, 41, 29, 19)$.

Similar results hold for the case when λ is symmetric and $D(\lambda)$ is concentrated at $\gamma = \gamma^* = \frac{p-1}{2}$.

Theorem 5.5. Suppose λ is a symmetric partition with empty p -core and let $D(\lambda)$ be concentrated at $\gamma = \gamma^* = \frac{p-1}{2}$. Then

- (1) $D'(\lambda)$ is a p -tuple of $p-1$ empty bisequences, with $D_{\frac{p-1}{2}}(\lambda) \neq \emptyset$ and $D_{\frac{p-1}{2}}(\lambda)_R$ and $D_{\frac{p-1}{2}}(\lambda)_L$ are of equal lengths.
- (2) $D_{\frac{p-1}{2}}(\lambda) = D(\lambda_{\frac{p-1}{2}})$.
- (3) Suppose $D(\lambda_{\frac{p-1}{2}}) = (w_1, \dots, w_\mu | w_1, \dots, w_\mu)$, and $D(\lambda_\gamma) = \emptyset$ when $\gamma \neq \frac{p-1}{2}$. Then

$$D(\lambda) = (z_1, \dots, z_\mu | z_1, \dots, z_\mu)$$

$$\text{where } z_i = \frac{p-1}{2} + w_i p.$$

Proof. By induction on $|\lambda|$. The minimal case is $|\lambda| = p$. In this case $D(\lambda) = (\frac{p-1}{2} | \frac{p-1}{2})$ and $D_{\frac{p-1}{2}}(\lambda) = (0|0)$. The remainder of the proof is similar to that of Theorem 5.2. \square

Corollary 5.6. Suppose λ is symmetric with empty p -core, such that λ is concentrated at $\{\frac{p-1}{2}\}$. Then $\delta(\lambda) = \cup_i \{(2m_i + 1)p\}$ if $\delta(\lambda_{\frac{p-1}{2}}) = \cup_i \{2m_i + 1\}$ for every $(m_i | m_i) \in D(\lambda_{\frac{p-1}{2}})$.

Proof. This follows from Theorem 2, part 3. \square

Example 5.7. Let $p = 5$. Suppose the 5-quotient is concentrated at $\{2\}$ and $\lambda_2 = (2^2)$. Then $D_2(\lambda) = (1, 0|1, 0)$ and $D(\lambda) = (7, 2|7, 2)$, $\delta(\lambda) = (15, 5)$.

6. SYMMETRIC PARTITIONS WITH AN EMPTY p -CORE

Now suppose λ is symmetric and has empty p -core. Fix a γ between 0 and $\frac{p-1}{2}$. Suppose $D(\lambda_{[\gamma]}) \subseteq D(\lambda)$ is the bisequence whose p -quotient $D'(\lambda_{[\gamma]})$ has just the components $D_\gamma(\lambda)$ and $D_{\gamma^*}(\lambda)$. Let $\lambda_{[\gamma]}$ be the symmetric partition corresponding to $D(\lambda_{[\gamma]})$. By Lemma 5.1,

$$\lambda_{[0]}, \dots, \lambda_{[\frac{p-1}{2}]}$$

have disjoint diagonals. Thus $\lambda_{[0]} + \lambda_{[1]} + \dots + \lambda_{[\frac{p-1}{2}]}$ is defined in the sense described in the remark before Theorem 5.2.

Theorem 6.1. *Suppose λ is symmetric and has empty p -core. Then*

$$\lambda = \lambda_{[0]} + \lambda_{[1]} + \dots + \lambda_{[\frac{p-1}{2}]}$$

$$D(\lambda) = \coprod_{1 \leq \gamma \leq \frac{p-1}{2}} D(\lambda_{[\gamma]})$$

Proof. By Lemma 5.1, it is clear that $D(\lambda_{[\gamma]}) \cap D(\lambda_{[\mu]}) = \emptyset$ when $\gamma \neq \mu$. Let $k_\gamma = |D_\gamma(\lambda)_R|$. By Theorem 5.2 and Theorem 5.5, for a fixed γ , we have for all $1 \leq i \leq k_\gamma$ and $1 \leq j \leq t$, the diagonal hooks of λ corresponding to $(\alpha'_{\gamma,i}|\alpha'_{\gamma,i})$, $(\alpha''_{\gamma,i}|\alpha''_{\gamma,i})$ and $(z_j|z_j)$ in $D(\lambda)$ have distinct lengths. Hence $\bigcap_{1 \leq \gamma \leq \frac{p-1}{2}} D(\lambda_{[\gamma]}) = \emptyset$. Since these exhaust the diagonal hooks arising from the p -quotient $D'(\lambda)$, and λ has an empty p -core, $\coprod_{1 \leq \gamma \leq \frac{p-1}{2}} D(\lambda_{[\gamma]})$ constitute all of the diagonal hook lengths of λ . \square

Example 6.2.

Suppose $p = 5$, $\lambda \vdash 190$ is symmetric with empty p -core and $\lambda_0 = (6^2, 2)$, $\lambda_1 = (3)$, $\lambda_2 = (2^2)$, $\lambda_3 = (1^3)$, $\lambda_4 = (3^2, 2^3)$. Then $D_0(\lambda) = \{2, 1|5, 4\}$, $D_1(\lambda) = \{0|2\}$, $D_2(\lambda) = \{1, 0|1, 0\}$, $D_3(\lambda) = \{2|0\}$, and $D_4(\lambda) = \{5, 4|2, 1\}$. Hence

$$D(\lambda) = \{25, 20, 14, 11, 9, 7, 3, 2|25, 20, 14, 11, 9, 7, 3, 2\}$$

and $\delta(\lambda) = (51, 41, 29, 23, 19, 15, 7, 5)$.

7. SYMMETRIC p -CORES

For any partition λ , let

$$D(\lambda)_{L,\gamma} = \{\alpha \in D(\lambda)_L : \alpha \equiv \gamma \pmod{p}\}$$

and

$$D(\lambda)_{R,\gamma} = \{\beta \in D(\lambda)_R : \beta \equiv \gamma \pmod{p}\}.$$

Let λ^0 be a symmetric p -core partition. Let $D(\lambda)_\gamma$ be the set of $(\beta|\beta) \in D(\lambda)$ such that $\beta \equiv \gamma \pmod{p}$. Then, in particular, $D(\lambda^0) = \cup_\gamma D(\lambda^0)_\gamma$ and $D(\lambda^0)_{\gamma'} \cap D(\lambda^0)_\gamma = \emptyset$ for $\gamma \neq \gamma'$.

Proposition 7.1. Suppose λ^0 is a symmetric p -core partition. Then for $\gamma \neq \gamma^*$,

$$D(\lambda^0)_\gamma \neq \emptyset \text{ implies } D(\lambda^0)_{\gamma^*} = \emptyset.$$

Proof. Suppose $(\alpha_i|\alpha_i) \in D(\lambda^0)_\gamma$ and $(\beta_j|\beta_j) \in D(\lambda^0)_{\gamma^*}$. Then in the notation of Proposition 3.3 we have $y'_i \equiv \theta(\lambda^0) + \frac{1}{2} + \gamma \pmod{p}$ and $y_j \equiv \theta(\lambda^0) - \frac{1}{2} - \gamma^* \pmod{p}$. But $\gamma^* = -1 - \gamma \pmod{p}$. Thus $y'_i - y_j \equiv \pmod{p}$, contradicting the assumption λ^0 is a p -core. \square

A symmetric λ is γ -packed if $D(\lambda)_\gamma$ consists of the elements $(\gamma + ip|\gamma + ip)$ for $i = 0, 1, \dots, r$. Let $X_{\gamma,+}$ be the subset of X_+ consisting of elements y' where

$$y' - \theta(\lambda) - \frac{1}{2} \equiv \gamma \pmod{p}.$$

We define $X_{\gamma,-}$ similarly.

Proposition 7.2. Suppose λ^0 is a symmetric p -core and $D(\lambda)_\gamma \neq \emptyset$. Then λ^0 is γ -packed.

Proof. Clearly if λ^0 is not γ -packed, then there exist integers y', z' , greater than $\theta(\lambda^0)$ such that $z' \equiv y' \pmod{p}$, $z' \notin X_+$, and $y' \in X_+$. In particular, $(z', y']$ is a p -hook of λ^0 . \square

Corollary 7.3. (symmetric p -core criterion) Let λ^0 be a symmetric partition. Then λ^0 is a p -core if and only if for every $\gamma \in \{0, \dots, p-1\}$ $D(\lambda^0)_\gamma \neq \emptyset$ implies that λ^0 is γ -packed and $D(\lambda^0)_{\gamma^*} = \emptyset$.

Proof. Clearly, if λ^0 is a p -core the result follows by Proposition 7.1 and Proposition 7.2. Suppose that for each $\gamma \in \{0, \dots, p-1\}$, if $D(\lambda^0)_\gamma \neq \emptyset$ then λ^0 is γ -packed and $D(\lambda^0)_{\gamma^*} = \emptyset$, but that λ^0 is not a p -core. Then, for some γ there exists a hook $h = (x, y']$ where $y' = \gamma + mp$, $x = \gamma + (m-1)p$. By symmetry, we can assume that $y' > \theta(\lambda^0)$. If $x > \theta(\lambda)$, then λ is not γ -packed, which is a contradiction. Now suppose $x < \theta(\lambda^0)$, then by symmetry there exists $x' > \theta(\lambda^0)$ such that

$x' - \theta(\lambda^0) - \frac{1}{2} \equiv \gamma^* \pmod{p}$. This implies $D(\lambda^0)_{\gamma^*} \neq \emptyset$, which is a contradiction. \square

Example 7.4.

Suppose $p = 5$ and $\lambda' \vdash 324$ such that λ' is symmetric and

$$\delta(\lambda') = (69, 59, 49, 39, 29, 27, 19, 17, 9, 7).$$

In particular, $D(\lambda')_R = (34, 29, 24, 19, 14, 13, 9, 8, 4, 3)$. Hence $D(\lambda')_{R,4} = (34, 29, 24, 19, 14, 9, 4)$ and $D(\lambda')_{R,3} = (13, 8, 3)$. Hence λ' is both 4-packed and 3-packed. Since $D(\lambda')_0 = \emptyset$ and $D(\lambda')_1 = \emptyset$, λ' is a 5-core by Theorem 7.3.

8. SYMMETRIC PARTITIONS WITH A NON-EMPTY p -CORE

We extend the results of Section 6 to the case of a symmetric partition with a non-empty p -core. Let $\bar{\lambda}$ be the symmetric partition that shares the p -quotient with λ , but has empty p -core. Hence $(\bar{\lambda})^0 = \emptyset$ and $(\bar{\lambda})_\gamma = \lambda_\gamma$ for $0 \leq \gamma \leq p-1$.

Now consider a symmetric partition λ of n with a non-empty p -core λ^0 . Let \bar{X} and X^0 be β -sets of $\bar{\lambda}$ and λ^0 respectively. Since $\lambda^0 \neq \emptyset$, we have $(X^0)_{\gamma,+} \neq \emptyset$, for some γ . Then $|D(\lambda^0)_\gamma| \neq \emptyset$. In particular, $|D(\lambda^0)_\gamma| = d_\gamma^0$ by Proposition 7.1. By the definition of $D'(\lambda)$ each $(\alpha|\alpha) \in D(\lambda^0)_\gamma$ contributes an element to both $D_{\gamma^*}(\lambda)_L$ and $D_\gamma(\lambda)_R$. ($D(\lambda^0)$ contributes nothing to $D_{\gamma^*}(\lambda)_R$ and $D_\gamma(\lambda)_L$.) The definition of $D'(\lambda)$ (and Proposition 7.1) forces $|D_\gamma(\lambda)_R| - |D_\gamma(\lambda)_L| = d_\gamma^0$. This implies $D'(\lambda)$ is not a p -tuple of bisequences. Specifically, $D_\gamma(\lambda) \neq D(\lambda_\gamma)$.

Define $\Omega' \subset \{0, \dots, p-1\}$ so that $\gamma' \in \Omega'$ if $D_{\gamma'}(\lambda^0) \neq \emptyset$ (i.e. $d_{\gamma'}^0 > 0$). Let $(\Omega')^* = \{p - \gamma' - 1 : \gamma' \in \Omega'\}$ and $U = \Omega' \cup (\Omega')^*$. Define $\Omega'' = \{0, \dots, p-1\} - U$.

Lemma 8.1.

- (1) $\theta(\lambda_{\gamma''}) = \theta(\bar{\lambda}_{\gamma''})$
- (2) $\theta(\lambda_{\gamma'}) = \theta(\bar{\lambda}_{\gamma'}) + d_{\gamma'}^0$

Proof. When all beads on all the runners of the abacus of λ are moved up completely one obtains the abacus diagram for λ^0 (Theorem 2.7.16, [2]). Since $D(\lambda^0)_{\frac{p-1}{2}}$ is empty, $d_{\frac{p-1}{2}}^0 = 0$ and the $\frac{p-1}{2}$ th runner of λ^0 is unchanged from the $\frac{p-1}{2}$ th runner of $\bar{\lambda}^0$. Let \bar{X}^0 and X^0 be the β -sets for λ^0 and $\bar{\lambda}^0$. Matching the $\frac{p-1}{2}$ th runners of λ^0 and $\bar{\lambda}^0$ one can superimpose the abacus of λ^0 onto of the abacus of $\bar{\lambda}^0$. It follows

that $|\bar{X}_{\gamma'}^0| + d_{\gamma'}^0 = |X_{\gamma'}^0|$ for $\gamma' \in \Omega'$. Also, $|\bar{X}_{\gamma''}^0| = |X_{\gamma''}^0|$ since $d_{\gamma''}^0 = 0$ for $\gamma'' \in \Omega''$. Hence $\theta(\lambda_{\gamma'}^0) = \theta(\bar{\lambda}_{\gamma'}^0) + d_{\gamma'}^0$ and $\theta(\lambda_{\gamma''}^0) = \theta(\bar{\lambda}_{\gamma''}^0)$. The result follows since $\theta(\lambda_{\gamma'}^0) = \theta(\bar{\lambda}_{\gamma'}^0)$ and $\theta(\lambda_{\gamma''}^0) = \theta(\bar{\lambda}_{\gamma''}^0)$ by Proposition 3.2. \square

We can describe $X_{\gamma'}$ using $\bar{X}_{\gamma'}$ and Lemma 8.1 in the following three steps which we call the $d_{\gamma'}^0$ -shift of $\bar{X}_{\gamma'}$.

- (1) $m_{\sigma} \in X_{\gamma',+}$ if $m_{\sigma} - d_{\gamma'}^0 > \theta(\bar{\lambda}_{\gamma'})$ and $m_{\sigma} - d_{\gamma'}^0 \in \bar{X}_{\gamma',+}$
- (2) $m_s \in X_{\gamma',+}$ if $\theta(\bar{\lambda}_{\gamma'}) < m_s < \theta(\bar{\lambda}_{\gamma'}) + d_{\gamma'}^0$ and $m_s - d_{\gamma'}^0 \notin \bar{X}_{\gamma',-}$
- (3) $m_t \in X_{\gamma',-}$ if $m_t - d_{\gamma'}^0 < \theta(\bar{\lambda}_{\gamma'})$.

Now consider the following sets

$$\begin{aligned} S_{\gamma'}(\bar{\lambda})_L &=: \{s : s \in \mathbb{N}, s \notin D_{\gamma'}(\bar{\lambda})_L, 0 \leq s \leq d_{\gamma'}^0 - 1\} \\ T_{\gamma'}(\bar{\lambda})_L &=: \{t : t \in D_{\gamma'}(\bar{\lambda})_L, t \geq d_{\gamma'}^0\}. \end{aligned}$$

Following the comments after Proposition 3.8, $S_{\gamma'}(\bar{\lambda})_L$ and $T_{\gamma'}(\bar{\lambda})_L$ are in bijection with the subsets of $\bar{X}_{\gamma'}$ in steps (2) and (3) of the definition of the $d_{\gamma'}^0$ -shift. Now we can interpret $D_{\gamma'}(\lambda)$ via the $d_{\gamma'}^0$ -shift of $\bar{X}_{\gamma'}$.

Proposition 8.2. $D_{\gamma'}(\lambda)$ is obtained from $D_{\gamma'}(\bar{\lambda})$ in the following three steps.

- (1) Each $\sigma \in D_{\gamma'}(\bar{\lambda})_R$ is sent to $\sigma + d_{\gamma'}^0 \in D_{\gamma'}(\lambda)_R$
- (2) Each $s \in S_{\gamma'}(\bar{\lambda})_L$ is sent to $d_{\gamma'}^0 - s - 1 \in D_{\gamma'}(\lambda)_R$
- (3) Each $t \in T_{\gamma'}(\bar{\lambda})_L$ is sent to $t - d_{\gamma'}^0 \in D_{\gamma'}(\lambda)_L$.

Proof. We prove part (2). Let $x_s = m_s - d_{\gamma'}^0$. Each $x_s < \theta(\bar{\lambda}_{\gamma'})$ where $x_s \notin \bar{X}_{\gamma',-}$ where $x_s + d_{\gamma'}^0 > \theta(\bar{\lambda}_{\gamma'})$ corresponds to some $s \in S_{\gamma'}(\bar{\lambda})_L$. Hence we have $\gamma + (x_s + d_{\gamma'}^0)p \in X_+$ by the usual p -quotient. Again, by Proposition 3.3,

$$\theta(\lambda) + \frac{1}{2} + \beta = \gamma + (x_s + d_{\gamma'}^0)p$$

for some $\beta \in D(\lambda)_R$. By substitution,

$$\beta = \gamma - \theta(\lambda) - \frac{1}{2} + (\theta(\bar{\lambda}_{\gamma'}) - \frac{1}{2})p + (d_{\gamma'}^0 - s)p.$$

By Lemma 3.7 we have

$$\beta = \gamma + (d_{\gamma'}^0 - s - 1)p.$$

By definition of $D'(\lambda)$, $x_s + d_{\gamma'}^0$ corresponds to $d_{\gamma'}^0 - \bar{s} - 1 \in D_{\gamma'}(\bar{\lambda})_R$.

The proofs of (1) and (3) are similar. \square

Proposition 8.3. Given λ^0 and $\bar{\lambda}$,

$$D(\lambda)_R = O_1 \cup O_2 \cup O_3 \cup O_4$$

where

$$\begin{aligned} O_1 &= \cup_{\gamma' \in \Omega'} \{ \gamma' + (\sigma + d_{\gamma'}^0)p : \sigma \in D_{\gamma'}(\bar{\lambda})_R \} \\ O_2 &= \cup_{\gamma' \in \Omega'} \{ \gamma' + (d_{\gamma'}^0 - s - 1)p : s \in S_{\gamma'}(\bar{\lambda}) \} \\ O_3 &= \cup_{\gamma' \in \Omega'} \{ (p - 1 - \gamma') + (t - d_{\gamma'}^0)p : t \in T_{\gamma'}(\bar{\lambda}) \} \\ O_4 &= \cup_{\gamma'' \in \Omega''} \{ \gamma' + \mu p : \mu \in D_{\gamma''}(\bar{\lambda})_R \}. \end{aligned}$$

Proof. This follows from the definition of $D'(\lambda)$ and Proposition 8.2. \square

Theorem 8.4. Given λ^0 and $\bar{\lambda}$,

$$\delta(\lambda) = \mathbb{O}_1 \cup \mathbb{O}_2 \cup \mathbb{O}_3 \cup \mathbb{O}_4$$

where

$$\begin{aligned} \mathbb{O}_1 &= \cup_{\gamma' \in \Omega'} \{ 2(\gamma' + (\sigma + d_{\gamma'}^0)p) + 1 : \sigma \in D_{\gamma'}(\bar{\lambda})_R \} \\ \mathbb{O}_2 &= \cup_{\gamma' \in \Omega'} \{ 2(\gamma' + (d_{\gamma'}^0 - s - 1)p) + 1 : s \in S_{\gamma'}(\lambda)_L \} \\ \mathbb{O}_3 &= \cup_{\gamma' \in \Omega'} \{ 2((p - 1 - \gamma') + (t - d_{\gamma'}^0)p) + 1 : t \in T_{\gamma'}(\lambda)_L \}. \\ \mathbb{O}_4 &= \cup_{\gamma'' \in \Omega''} \{ 2(\gamma' + \mu p) + 1 : \mu \in D_{\gamma''}(\bar{\lambda})_R \} \end{aligned}$$

where $d_{\gamma'}^0 = |D(\lambda^0)_{\gamma'}|$, $\gamma' \in \Omega'$ and $D_{\gamma'}(\lambda)_R$, $S_{\gamma'}(\lambda)_L$ and $T_{\gamma'}(\lambda)_L$ are as above.

Proof. Follows from the Proposition 8.3 and the relationship between $D(\lambda)$ and $\delta(\lambda)$. \square

[Note: In the case $d_{\gamma}^0 = 0$ for all $0 \leq \gamma \leq p - 1$, Proposition 8.3 reverts to Theorem 6.1.]

Example 8.5.

Suppose $p = 5$ and $\eta \vdash 514$ such that η is symmetric such that $\eta^0 = \lambda'$, where λ' is as in Example 7.4. Furthermore, let $\eta_i = \lambda_i$ for $0 \leq i \leq p - 1$, where λ_i is as in Example 6.2. In this case we have $d_3^0 = 3$, $d_4^0 = 7$, $D_3(\bar{\eta}) = \{2|0\}$, and $D_4(\bar{\eta}) = \{5, 4|2, 1\}$. Hence, by Proposition 8.2, we have

$$\begin{aligned} D_2(\eta)_R &= \{1, 0\} \\ D_2(\eta)_L &= \{1, 0\} \\ D_3(\eta)_R &= \{2, 1, 0\} \\ D_3(\eta)_L &= \emptyset \\ D_4(\eta)_R &= \{9, 8, 6, 5, 4, 3, 0\} \\ D_4(\eta)_L &= \emptyset. \end{aligned}$$

Then by Proposition 8.3,

$$\begin{aligned} D(\eta)_{R,2} &= \{6, 2\} \\ D(\eta)_{R,3} &= \{13, 8, 3\} \\ D(\eta)_{L,3} &= \emptyset \\ D(\eta)_{R,4} &= \{49, 44, 34, 29, 24, 19, 4\} \\ D(\eta)_{L,4} &= \emptyset. \end{aligned}$$

Finally, by Theorem 8.4 we have

$$\delta(\eta) = (99, 69, 59, 49, 39, 37, 27, 17, 13, 9, 7, 5).$$

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